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On the Construction and Convergence of Multivariate Interpolation Operators

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In recent years several multivariate interpolation methods have been investigated. The main interest has been concentrated on *existence and uniqueness theorems* (see, e.g., Milne *et al.* [13], Thacher [20], and Thacher and Milne [21]) as well as on *convergence properties* (cf. Amelkovic [1], Moldovan [14], Shisha and Mond [17], Sloss [18], Haussmann and Knoop [11] and [9]) of certain interpolation processes. The authors mentioned above used specific methods to get results in particular cases.

It is the purpose of the present paper to provide a unified theory to deal with multivariate interpolation processes. This theory is based on the use of tensor products. In the first section we briefly consider existence, uniqueness, and representation properties of multivariate interpolation. For convergence statements we need some investigations on d-fold cross norms established in Section 2. Then we give convergence theorems (including quantitative assertions) for interpolation operators in Section 3. These results are based on the knowledge of certain one-dimensional theorems and on the structure of the multivariate interpolation process only. In Section 4 we illustrate our results by several examples, including cases of equicontinuous and non-equicontinuous sequences of interpolation operators.

1. CONSTRUCTION OF MULTIVARIATE INTERPOLATION OPERATORS

Let $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, X a (real or complex) vector space, U an $(n+1)$-dimensional subspace of X, and $\phi_0, \phi_1, ..., \phi_n$ linear functionals, i.e., $\phi_\nu \in X^*$, the algebraic dual of X $(0 \leqslant \nu \leqslant n)$. Then we have the following *interpolation problem* $\mathscr{P} := (X, U; \phi_0, ..., \phi_n)$: Given an arbitrary $x \in X$, does there exist a $u \in U$, such that the interpolation conditions

$$\phi_\nu(u) = \phi_\nu(x) \qquad (0 \leqslant \nu \leqslant n) \tag{1}$$

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are satisfied? If for any $x \in X$ the interpolating element $u \in U$ is determined uniquely, we shall call $\mathscr{P}$ an *interpolation system*. This is the case if and only if the restrictions $\phi_\nu \mid U$ $(0 \leqslant \nu \leqslant n)$ form an (algebraic) basis of U^* (see Davis [5]).

Given any interpolation system $\mathscr{P} = (X, U; \phi_0, ..., \phi_n)$ there exists a uniquely determined interpolation operator $P: X \to U$ which maps an $x \in X$ onto its interpolating element u. P can be constructed as follows: There is a unique dual basis $\{u_0, ..., u_n\}$ of U with respect to $\{\phi_0 \mid U, ..., \phi_n \mid U\}$ such that

$$\phi_\nu(u_\mu) = \delta_{\mu\nu} \qquad 0 \leqslant \mu, \nu \leqslant n.$$

With the aid of these elements we define the interpolation operator P by

$$P(x) = \sum_{\nu=0}^{n} \phi_\nu(x) \cdot u_\nu .$$

Obviously, (1) is satisfied for $u = Px$, and P is a linear and idempotent operator.

Now let $d \in \mathbb{N}$, $d \geqslant 2$. We are going to define d-dimensional interpolation problems out of one-dimensional ones by means of tensor products.

DEFINITION 1. Let $n_\delta \in \mathbb{N}_0$ and the interpolation problems

$$\mathscr{P}_\delta := (X_\delta, U_\delta; \phi_{\delta 0}, ..., \phi_{\delta n_\delta})$$

be given for $1 \leqslant \delta \leqslant d$. Then

$$\begin{aligned}\mathscr{P} &:= \bigotimes_{1\leqslant\delta\leqslant d} \mathscr{P}_\delta \\ &:= \Big(\bigotimes_{1\leqslant\delta\leqslant d} X_\delta, \bigotimes_{1\leqslant\delta\leqslant d} U_\delta; \phi_{1\nu_1} \otimes \cdots \otimes \phi_{d\nu_d}: 0 \leqslant \nu_\delta \leqslant n_\delta, 1 \leqslant \delta \leqslant d\Big)\end{aligned}$$

is called the tensor product of the given problems $\mathscr{P}_\delta$ $(1 \leqslant \delta \leqslant d)$. In certain situations it is convenient to replace $\otimes_{1\leqslant\delta\leqslant d} X_\delta$ by a larger vector space $Z \supset \otimes_{1\leqslant\delta\leqslant d} X_\delta$ and the functionals $\phi_{1\nu_1} \otimes \cdots \otimes \phi_{d\nu_d}$ by certain extensions to Z.

The following theorem answers the question whether the tensor product of interpolation systems is an interpolation system again.

THEOREM 1. (cf., [8]). *Let the interpolation problems* $\mathscr{P}_\delta := (X_\delta, U_\delta;$ $\phi_{\delta 0}, ..., \phi_{\delta n_\delta})$ *be given.* $\mathscr{P} := \otimes_{1\leqslant\delta\leqslant d} \mathscr{P}_\delta$ *(and any extension of* $\mathscr{P}$*) is an interpolation system if and only if* $\mathscr{P}_1, ..., \mathscr{P}_d$ *are interpolation systems.*

Now we are going to construct the interpolation operator corresponding to the tensor product $\mathscr{P}$ of the interpolation systems $\mathscr{P}_\delta := (X_\delta, U_\delta; \phi_{\delta 0}, ..., \phi_{\delta n_\delta})$ $(1 \leqslant \delta \leqslant d)$. Let $\{u_{\delta 0}, ..., u_{\delta n_\delta}\}$ be the dual basis of $\{\phi_{\delta 0} \mid U_\delta, ..., \phi_{\delta n_\delta} \mid U_\delta\}$. It is easy to see that the interpolation operator corresponding to $\mathscr{P} := \bigotimes_{1 \leqslant \delta \leqslant d} \mathscr{P}_\delta$ has the following form:

$$P = \bigotimes_{1 \leqslant \delta \leqslant d} P_\delta = \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_d=0}^{n_d} (\phi_{1\nu_1} \otimes \cdots \otimes \phi_{d\nu_d}) \cdot (u_{1\nu_1} \otimes \cdots \otimes u_{d\nu_d}).$$

Examples of certain multivariate interpolation operators will be considered in Section 4.

2. d-Fold Cross Norms

In order to get convergence statements on multivariate interpolation operators we have to provide the tensor product spaces with appropriate cross norms.

Definition 2. (cf., Schatten [16]). Let $(X_\delta, \|\cdot\|_\delta)$ as well as $(Y_\delta, |\cdot|_\delta)$ be (real or complex) normed vector spaces $(1 \leqslant \delta \leqslant d)$.

(i) A norm η on the d-fold tensor product $\bigotimes_{1 \leqslant \delta \leqslant d} X_\delta$ is called a *d-fold cross-norm* if for any element of the form $x_1 \otimes \cdots \otimes x_d$ $(x_\delta \in X_\delta$, $1 \leqslant \delta \leqslant d)$ the following identity holds:

$$\eta(x_1 \otimes \cdots \otimes x_d) = \prod_{\delta=1}^{d} \|x_\delta\|_\delta .$$

If η is a d-fold cross norm on $\bigotimes_{1 \leqslant \delta \leqslant d} X_\delta$ then the normed vector space $(\bigotimes_{1 \leqslant \delta \leqslant d} X_\delta, \eta)$ will be designated briefly by $\bigotimes^{\eta}_{1 \leqslant \delta \leqslant d} X_\delta$. In addition, its completion will be denoted by $\hat{\bigotimes}^{\eta}_{1 \leqslant \delta \leqslant d} X_\delta$.

(ii) Let η and ω be d-fold cross norms on $\bigotimes_{1 \leqslant \delta \leqslant d} X_\delta$ resp., $\bigotimes_{1 \leqslant \delta \leqslant d} Y_\delta$. (η, ω) will be called *uniform cross norms with respect to* $(\bigotimes_{1 \leqslant \delta \leqslant d} X_\delta$, $\bigotimes_{1 \leqslant \delta \leqslant d} Y_\delta)$ if for any continuous linear operators $S_\delta : X_\delta \to Y_\delta$ $(1 \leqslant \delta \leqslant d)$ the tensor product operator

$$\bigotimes_{1 \leqslant \delta \leqslant d} S_\delta \colon \bigotimes^{\eta}_{1 \leqslant \delta \leqslant d} X_\delta \to \bigotimes^{\omega}_{1 \leqslant \delta \leqslant d} Y_\delta$$

is also continuous, and, if in addition, using the induced operator norms $\|\cdot\|$ and $\|\cdot\|_{[\delta]}$, we have

$$\Big\| \bigotimes_{1 \leqslant \delta \leqslant d} S_\delta \Big\| = \prod_{\delta=1}^{d} \| S_\delta \|_{[\delta]} .$$

In the special case when we have $X_\delta = Y_\delta$ for $1 \leqslant \delta \leqslant d$ and $\eta = \omega$ then η is called a uniform cross norm on $\bigotimes_{1\leqslant\delta\leqslant d} X_\delta$ (cf., [16]).

Now we are going to consider special d-fold cross norms on the (algebraic) tensor product $\bigotimes_{1\leqslant\delta\leqslant d} X_\delta$ of the normed vector spaces $(X_\delta, \|\cdot\|_\delta)$. To this end let $\Xi \in \bigotimes_{1\leqslant\delta\leqslant d} X_\delta$ be an arbitrary element and

$$\sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_d=0}^{n_d} \xi_{\nu_1\cdots\nu_d} \cdot x_{1\nu_1} \otimes \cdots \otimes x_{d\nu_d}$$

a representation of Ξ. Then we define

(1) the *d-fold ϵ-norm* (cf., [23]): Let $(X_\delta', \|\cdot\|_{(\delta)})$ be the normed dual of $(X_\delta, \|\cdot\|_\delta)$ for $1 \leqslant \delta \leqslant d$. Then we have

$$\epsilon(\Xi) := \sup_{\substack{\phi_1 \in X_1' \\ \|\phi_1\|_{(1)}\leqslant 1}} \cdots \sup_{\substack{\phi_d \in X_d' \\ \|\phi_d\|_{(d)}\leqslant 1}} \left| \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_d=0}^{n_d} \xi_{\nu_1\cdots\nu_d} \cdot \prod_{\delta=1}^{d} \phi_\delta(x_{\delta\nu_\delta}) \right|.$$

It turns out that the value $\epsilon(\Xi)$ does not depend on the particular representation of $\Xi \in \bigotimes_{1\leqslant\delta\leqslant d} X_\delta$;

(2) the *d-fold π-norm* (cf., [23]):

$$\pi(\Xi) := \inf \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_d=0}^{n_d} | \xi_{\nu_1\cdots\nu_d} | \cdot \prod_{\delta=1}^{d} \| x_{\delta\nu_\delta} \|_\delta ,$$

where the infimum is to be taken over all equivalent representations of Ξ. Therefore, $\pi(\Xi)$ does not depend on the specific representation of Ξ;

(3) the *d-fold pre-Hilbert space norm* α (cf., [6]): Here we have to suppose that the spaces $(X_\delta, \|\cdot\|_\delta)$ are pre-Hilbert spaces with the inner products $(\cdot \mid \cdot)_\delta$ $(1 \leqslant \delta \leqslant d)$. Then the α-norm is defined by

$$\alpha(\Xi) := \left\{ \sum_{\mu_1=0}^{n_1} \sum_{\nu_1=0}^{n_1} \cdots \sum_{\mu_d=0}^{n_d} \sum_{\nu_d=0}^{n_d} \xi_{\mu_1\cdots\mu_d} \cdot \xi_{\nu_1\cdots\nu_d} \cdot \prod_{\delta=1}^{d} (x_{\delta\mu_\delta} \mid x_{\delta\nu_\delta})_\delta \right\}^{1/2}.$$

In this case, too, $\alpha(\Xi)$ is independent of the representation of $\Xi \in \bigotimes_{1\leqslant\delta\leqslant d} X_\delta$ (cf., [16]).

THEOREM 2. *Let $(X_\delta, \|\cdot\|_\delta)$ and $(Y_\delta, |\cdot|_\delta)$ be normed vector spaces (resp., pre-Hilbert spaces) for $1 \leqslant \delta \leqslant d$. Then (ϵ, ϵ) and (π, π) (resp., (α, α)) are uniform cross norms with respect to $(\bigotimes_{1\leqslant\delta\leqslant d} X_\delta, \bigotimes_{1\leqslant\delta\leqslant d} Y_\delta)$. Furthermore, for $\tau = \epsilon, \pi$ (resp., α) the spaces*

$$X_1 \widehat{\otimes}_{\tau_2} X_2 \widehat{\otimes}_{\tau_2} \cdots \widehat{\otimes}_{\tau_2} X_d \qquad \text{and} \qquad \widehat{\bigotimes_{1\leqslant\delta\leqslant d}}^{\tau_d} X_\delta \tag{2}$$

are isometric isomorphic independent of any brackets used when taking tensor *products and completions in the first space in* (2). *Here and in the following, if necessary,* τ (*resp.*, ϵ, π, α) *are provided with an index* k ($2 \leqslant k \leqslant d$) *to note that the occuring cross norms are k-fold ones.*

Proof. We are going to show that the spaces

$$\begin{gathered} X_1 \otimes_{\tau_2} (X_2 \otimes_{\tau_2} (\cdots (X_{d-1} \otimes_{\tau_2} X_d) \cdots)), \\ ((\cdots (X_1 \otimes_{\tau_2} X_2) \cdots) \otimes_{\tau_2} X_{d-1}) \otimes_{\tau_2} X_d \end{gathered} \tag{3}$$

and $\otimes_{1 \leqslant \delta \leqslant d}^{\tau_d} X_\delta$ coincide for $\tau = \epsilon, \pi$, and α. To do this we first prove that we have

$$X_1 \otimes_{\tau_2} \left(\bigotimes_{2 \leqslant \delta \leqslant d}^{\tau_{d-1}} X_\delta \right) = \bigotimes_{1 \leqslant \delta \leqslant d}^{\tau_d} X_\delta \tag{4}$$

as well as

$$\left(\bigotimes_{1 \leqslant \delta \leqslant d-1}^{\tau_{d-1}} X_\delta \right) \otimes_{\tau_2} X_d = \bigotimes_{1 \leqslant \delta \leqslant d}^{\tau_d} X_\delta \,. \tag{5}$$

For the special case $d = 3$, from (4) and (5) we deduce that forming normed tensor products is an associative operation for $\tau = \epsilon, \pi, \alpha$. By induction, we get that the spaces in (3) coincide, and that $X_1 \otimes_{\tau_2} X_2 \otimes_{\tau_2} \cdots \otimes_{\tau_2} X_d$ is independent of any bracketing. Since the completion of a normed vector space is uniquely determined, the statement will be settled if (4) and (5) can be verified. Because (5) can be proved in the same way as (4) it is sufficient to prove (4). We restrict our proof to the case $\tau = \epsilon$, the proofs for $\tau = \pi$ and α run in a similar way.

Let

$$h := \sum_{i=1}^{n} x_{1i} \otimes x_{2i} \otimes \cdots \otimes x_{di}$$

be an arbitrary element of $\otimes_{1 \leqslant \delta \leqslant d} X_\delta$, and $(X_\delta', \|\cdot\|_{(\delta)})$ the normed duals of the X_δ ($1 \leqslant \delta \leqslant d$).

Then

$$\| h \|_{X_1 \otimes_{\epsilon_2} (\otimes_{2 \leqslant \delta \leqslant d}^{\epsilon_{d-1}} X_\delta)} = \sup_{\substack{x_1' \in X_1' \\ \|x_1'\|_{(1)} \leqslant 1}} \sup_{\substack{\psi \in (\otimes_{2 \leqslant \delta \leqslant d}^{\epsilon_{d-1}} X_\delta)' \\ \|\psi\| \leqslant 1}} \left| \sum_{i=1}^{n} x_1'(x_{1i}) \cdot \psi(x_{2i} \otimes \cdots \otimes x_{di}) \right|.$$

For any $x_1' \in X_1'$ we have

$$h_{x_1'} := \sum_{i=1}^{n} x_1'(x_{1i}) \cdot (x_{2i} \otimes \cdots \otimes x_{di}) \in \bigotimes_{2 \leqslant \delta \leqslant d}^{\epsilon_{d-1}} X_\delta \,.$$

The Hahn–Banach theorem implies the validity of the following equation (cf., [19]):

$$\sup_{\substack{\psi \in (\otimes_{2\leqslant\delta\leqslant d}^{\epsilon_{d-1}} X_\delta)' \\ \|\psi\| \leqslant 1}} \left| \sum_{i=1}^{n} x_1'(x_{1i}) \cdot \psi(x_{2i} \otimes \cdots \otimes x_{di}) \right| = \| h_{x_1'} \|_{\epsilon_{d-1}} .$$

By the definition of the ϵ_{d-1}-norm we get

$$\| h_{x_1'} \|_{\epsilon_{d-1}} = \sup_{\substack{x_2' \in X_2' \\ \|x_2'\|_{(2)} \leqslant 1}} \cdots \sup_{\substack{x_d' \in X_d' \\ \|x_d'\|_{(d)} \leqslant 1}} \left| \sum_{i=1}^{n} x_1'(x_{1i}) \cdot x_2'(x_{2i}) \cdot \cdots \cdot x_d'(x_{di}) \right| .$$

Hence, for any $h \in \otimes_{1\leqslant\delta\leqslant d} X_\delta$ we have

$$\begin{aligned} \| h \|_{X_1 \otimes_{\epsilon_2} (\otimes_{2\leqslant\delta\leqslant d}^{\epsilon_{d-1}} X_\delta)} &= \sup_{\substack{x_1' \in X_1' \\ \|x_1'\|_{(1)} \leqslant 1}} \cdots \sup_{\substack{x_d' \in X_d' \\ \|x_d'\|_{(d)} \leqslant 1}} \left| \sum_{i=1}^{n} x_1'(x_{1i}) \cdot \cdots \cdot x_d'(x_{di}) \right| \\ &= \epsilon_d(h). \quad \blacksquare \end{aligned}$$

For the applications in Section 4 we need some knowledge on the ϵ-norm on the tensor product of certain normed vector spaces. To this end let $I_\delta = [a_\delta, b_\delta] \subset \mathbb{R}$ be nontrivial compact intervals, and $C(I_\delta)$ the vector space of all continuous real functions on I_δ provided with the Chebyshev (or sup-) norm

$$\| x \|_\delta := \sup_{t \in I_\delta} | x(t) | \qquad (x \in C(I_\delta),\ 1 \leqslant \delta \leqslant d).$$

Then we have the isometry (cf., [23]):

$$C\left(\bigtimes_{\delta=1}^{d} I_\delta \right) \cong \widehat{\bigotimes_{1\leqslant\delta\leqslant d}}^{\epsilon} C(I_\delta).$$

PROPOSITION 1. *Let the normed vector spaces $C(I_\delta)$ be given for $1 \leqslant \delta \leqslant d$ as well as $\delta_0 \in \{1, 2,..., d\}$, and let $L : C(I_{\delta_0}) \to C(I_{\delta_0})$ be a continuous linear mapping. Then for any $h \in C(\bigtimes_{\delta=1}^{d} I_\delta)$ the following equality holds*:

$$\sup_{\substack{x_\delta \in I_\delta \\ 1\leqslant\delta\leqslant d \\ \delta\neq\delta_0}} \| L(h_{\delta_0}) \|_{\delta_0} = \| (id^1 \widehat{\otimes} \cdots \widehat{\otimes}\, id^{\delta_0 - 1} \widehat{\otimes}\, L \,\widehat{\otimes}\, id^{\delta_0+1} \widehat{\otimes} \cdots \widehat{\otimes}\, id^d)(h) \|_\epsilon .$$

Here h_{δ_0} designates the δ_0th partial mapping

$$h_{\delta_0}\colon I_{\delta_0} \ni x \mapsto h(x_1, ..., x_{\delta_0 - 1}, x, x_{\delta_0+1}, ..., x_d) \in \mathbb{R}$$

with fixed points $x_\delta \in I_\delta$ $(\delta \neq \delta_0)$, and id^δ is the δth identity mapping id^δ: $C(I_\delta) \to C(I_\delta)$. By ^ we denote the uniquely determined continuous extensions *of tensor product operators to the completion of the corresponding tensor product space.*

Proof. Given an $h \in C(\mathsf{X}_{\delta=1}^{d} I_\delta)$, the mapping

$$h_L: (x_1, x_2, \ldots, x_d) \mapsto [L(h_{\delta_0})](x_{\delta_0})$$

is an element of $C(\mathsf{X}_{\delta=1}^{d} I_\delta)$. The mapping

$$H_L: C\left(\mathop{\mathsf{X}}_{\delta=1}^{d} I_\delta\right) \ni h \to h_L \in C\left(\mathop{\mathsf{X}}_{\delta=1}^{d} I_\delta\right)$$

is linear and continuous. On $\bigotimes_{1\leqslant\delta\leqslant d}^{\varepsilon} C(I_\delta)$ H_L coincides with

$$\tilde{H}_L := id^1 \otimes \cdots \otimes id^{\delta_0-1} \otimes L \otimes id^{\delta_0+1} \otimes \cdots \otimes id^d.$$

Therefore, H_L is the uniquely determined continuous extension of $\tilde{H}_L$ to the completion $\widehat{\bigotimes}_{1\leqslant\delta\leqslant d}^{\varepsilon} C(I_\delta)$. With the aid of this we get

$$\begin{aligned}\|(id^1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} id^{\delta_0-1} \mathbin{\widehat{\otimes}} L \mathbin{\widehat{\otimes}} id^{\delta_0+1} \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} id^d)(h)\|_\varepsilon \\ = \| H_L(h)\|_\varepsilon = \sup_{\substack{x_\delta\in I_\delta \\ 1\leqslant\delta\leqslant d}} |[L(h_{\delta_0})](x_{\delta_0})| = \sup_{\substack{x_\delta\in I_\delta \\ 1\leqslant\delta\leqslant d \\ \delta\neq\delta_0}} \| L(h_{\delta_0})\|_{\delta_0};\end{aligned}$$

hence, the proposition holds true. ∎

3. Convergence of Linear Interpolation Operators

Given a sequence $\mathscr{P}_n := \{X, U_n ; \phi_0^{(n)}, \ldots, \phi_n^{(n)}\}$ of interpolation systems, then corresponding to Section 1, there is defined a sequence $\{L_n\}_{n\in\mathbb{N}}$ of linear interpolation operators:

$$L_n : X \to U_n \subset X.$$

The theorems and definitions in this section will be given for sequences of general linear operators. As special cases we can get statements for sequences of interpolation operators. In addition, one can get corollaries for operators which do not arise from interpolation systems. As an example we mention multivariate Bernstein operators (see, e.g., [2]).

Definition 3. Let $(X, \|\cdot\|)$ be a normed vector space and $\{P_n\}_{n\in\mathbb{N}}$, a sequence of linear operators:

$$P_n : X \to X \qquad (n \in \mathbb{N}).$$

Further, let $\beta(n)$ be an arbitrary positive real sequence converging to zero. $\{P_n\}_{n\in\mathbb{N}}$ is called convergent of order $\beta(n)$ at the point $x \in X$ if we have

$$\| x - P_n x \| = O(\beta(n)) \qquad (n \to \infty).$$

For the pointwise convergence of sequences of tensor products of linear (not necessarily continuous) operators on the algebraic tensor product of normed vector spaces one gets the following generalization of a result in [7, 10]:

THEOREM 3. *Let $(X_\delta, \|\cdot\|_\delta)$ for $1 \leqslant \delta \leqslant d$ be normed vector spaces, $\{P_n\}_{n\in\mathbb{N}}$ sequences of endomorphisms of X_δ, η a d-fold cross norm on $\bigotimes_{1\leqslant\delta\leqslant d} X_\delta$, and*

$$h = \sum_{\mu=1}^{m} x_{1\mu} \otimes \cdots \otimes x_{d\mu} \in \bigotimes_{1\leqslant\delta\leqslant d} X_\delta .$$

Suppose the sequences $\{P_n{}^\delta\}_{n\in\mathbb{N}}$ converge for the elements $x_{\delta\mu}$ $(1 \leqslant \delta \leqslant d$, $1 \leqslant \mu \leqslant m)$ of the orders $\alpha_{\delta\mu}(n)$. Then

$$\bigotimes_{1\leqslant\delta\leqslant d} P_n{}^\delta \colon \bigotimes_{1\leqslant\delta\leqslant d}^{\eta} X_\delta \to \bigotimes_{1\leqslant\delta\leqslant d}^{\eta} X_\delta$$

converges of order $\alpha(n) := \max_{1\leqslant\delta\leqslant d, 1\leqslant\mu\leqslant m} \alpha_{\delta\mu}(n)$ at the given element h.

We are going to consider continuous tensor product interpolation operators on the completion $\widehat{\bigotimes}{}^{\eta}_{1\leqslant\delta\leqslant d} X_\delta$ of Banach spaces X_δ $(1 \leqslant \delta \leqslant d)$. First we get as a consequence of the Banach–Steinhaus theorem the following:

THEOREM 4. *Given the Banach spaces $(X_\delta, \|\cdot\|_\delta)$ for $1 \leqslant \delta \leqslant d$, and let $\{P_n{}^\delta\}_{n\in\mathbb{N}}$ be d sequences of continuous linear operators*

$$P_n{}^\delta : X_\delta \to X_\delta \qquad (1 \leqslant \delta \leqslant d)$$

such that $P_n{}^\delta x$ converges to x for all $x \in X_\delta$ $(1 \leqslant \delta \leqslant d$, $n \to \infty)$. Suppose, $\bigotimes_{1\leqslant\delta\leqslant d} X_\delta$ is provided with a uniform cross norm η. Then

$$\widehat{\bigotimes_{1\leqslant\delta\leqslant d}} P_n{}^\delta \colon \widehat{\bigotimes}{}^{\eta}_{1\leqslant\delta\leqslant d} X_\delta \to \widehat{\bigotimes}{}^{\eta}_{1\leqslant\delta\leqslant d} X_\delta$$

converges pointwise on $\widehat{\bigotimes}{}^{\eta}_{1\leqslant\delta\leqslant d} X_\delta$ to the identity mapping.

Theorem 4 describes the qualitative behavior of the convergence of continuous linear interpolation operators. In order to get assertions on the quantitative behavior for not necessarily equicontinuous linear operators we prove

THEOREM 5. *Let $(X_\delta, \|\cdot\|_\delta)$ be normed vector spaces $(1 \leqslant \delta \leqslant d)$, η a uniform cross norm on $\otimes_{1\leqslant\delta\leqslant d} X_\delta$, and $\{P_n^{\delta}\}_{n\in\mathbb{N}}$ sequences of continuous linear operators on X_δ $(1 \leqslant \delta \leqslant d)$. Then for an arbitrary $k \in \widehat{\otimes}^{\eta}_{1\leqslant\delta\leqslant d} X_\delta$ the following estimation holds true*:

$$\left\| k - \left(\widehat{\bigotimes_{1\leqslant\delta\leqslant d}} P_n^{\delta} \right)(k) \right\|_\eta \leqslant \min_{\sigma\in\mathscr{S}_d} \left\{ \sum_{\delta=1}^{d} \| k - Q_n^{\sigma(\delta)}(k)\|_\eta \cdot \prod_{\nu=\delta+1}^{d} \| P_n^{\sigma(\nu)} \|_{[\sigma(\nu)]} \right\}.$$

where $\mathscr{S}_d$ designates the symmetric group of all permutations of the elements $\{1, 2,\ldots, d\}$, and

$$Q_n^{\nu} := id^1 \widehat{\otimes} \cdots \widehat{\otimes}\, id^{\nu-1} \widehat{\otimes} P_n^{\nu} \widehat{\otimes}\, id^{\nu+1} \widehat{\otimes} \cdots \widehat{\otimes}\, id^d.$$

Proof. Let $\sigma \in \mathscr{S}_d$ be given. By means of well-known density arguments we get

$$\begin{aligned} &id^1 \widehat{\otimes}\, id^2 \widehat{\otimes} \cdots \widehat{\otimes}\, id^d - P_n^1 \widehat{\otimes} P_n^2 \widehat{\otimes} \cdots \widehat{\otimes} P_n^d \\ &\quad = id^1 \widehat{\otimes} \cdots \widehat{\otimes}\, id^{\sigma(d)-1} \widehat{\otimes} (id^{\sigma(d)} - P_n^{\sigma(d)}) \widehat{\otimes}\, id^{\sigma(d)+1} \widehat{\otimes} \cdots \widehat{\otimes}\, id^d \\ &\qquad - id^1 \widehat{\otimes} \cdots \widehat{\otimes} (id^{\sigma(d-1)} - P_n^{\sigma(d-1)}) \widehat{\otimes}\, id^{\sigma(d-1)+1} \widehat{\otimes} \cdots \widehat{\otimes} P_n^{\sigma(d)} \widehat{\otimes} \cdots \widehat{\otimes}\, id^d \\ &\qquad + \cdots + P_n^1 \widehat{\otimes} P_n^2 \widehat{\otimes} \cdots \widehat{\otimes} (id^{\sigma(1)} - P_n^{\sigma(1)}) \widehat{\otimes} P_n^{\sigma(1)+1} \widehat{\otimes} \cdots \widehat{\otimes} P_n^d. \end{aligned}$$

Since η is a uniform cross norm this latter relation yields

$$\begin{aligned} &\left\| k - \left(\widehat{\bigotimes_{1\leqslant\delta\leqslant d}} P_n^{\delta} \right)(k) \right\|_\eta \\ &\qquad \leqslant \sum_{\delta=1}^{d} \| k - (id^1 \widehat{\otimes} \cdots \widehat{\otimes} P_n^{\sigma(\delta)} \widehat{\otimes}\, id^{\sigma(\delta)+1} \widehat{\otimes} \cdots \widehat{\otimes}\, id^d)(k)\|_\eta \\ &\qquad\quad \cdot \prod_{\nu=\delta+1}^{d} \| P_n^{\sigma(\nu)} \|_{[\sigma(\nu)]} . \end{aligned} \tag{6}$$

This relation holds true for any permutation $\sigma \in \mathscr{S}_d$; hence, the assertion is proved. ∎

In the special case $d = 2$ this theorem can be found in [12].

Remark. The occurrence of the norms $\| P_n^{\sigma(\nu)} \|_{[\sigma(\nu)]}$ in (6) causes a deterioration of convergence in the multivariate case in comparison with the convergence of the "components"

$$\| k - (id^1 \widehat{\otimes} \cdots \widehat{\otimes} P_n^{\sigma(\delta)} \widehat{\otimes}\, id^{\sigma(\delta)+1} \widehat{\otimes} \cdots \widehat{\otimes}\, id^d)(k)\|_\eta .$$

Since we can choose an advantageous permutation $\sigma \in \mathscr{S}_d$ we can get an optimal estimation of the form (6).

If we restrict the continuous linear operators to appropriate subspaces, we get the same order of convergence in the multivariate case as in the one-dimensional components, which can be stated as follows:

THEOREM 6. *Let* $(Y_\delta, |\cdot|_\delta)$ *be normed vector spaces* $(1 \leqslant \delta \leqslant d)$, $\{P_n^\delta\}_{n\in\mathbb{N}}$ *sequences of continuous endomorphisms on* Y_δ $(1 \leqslant \delta \leqslant d)$. *Further let* X_δ *be subspaces of* Y_δ, *and let there exist a finer norm* $\|\cdot\|_\delta$ *on* X_δ *than the induced norm* (*i.e., suppose there exist real numbers* c_δ *such that* $|x|_\delta \leqslant c_\delta \|x\|_\delta$ *for all* $x \in X_\delta$ $(1 \leqslant \delta \leqslant d)$), *which makes the sequence* $\{\tilde{P}_n^\delta\}_{n\in\mathbb{N}}$ *defined by*

$$\tilde{P}_n^\delta : (X_\delta, \|\cdot\|_\delta) \ni x \mapsto P_n^\delta(x) \in (Y_\delta, |\cdot|_\delta)$$

uniformly convergent of order $\alpha_\delta(n)$ *to the* (*continuous linear*) *injection*

$$I^\delta : (X_\delta, \|\cdot\|_\delta) \to (Y_\delta, |\cdot|_\delta).$$

If (η, ω) *are uniform d-fold cross norms with respect to* $(\bigotimes_{1\leqslant\delta\leqslant d} X_\delta, \bigotimes_{1\leqslant\delta\leqslant d} Y_\delta)$ *then for any* $k \in \widehat{\bigotimes}{}^{\eta}_{1\leqslant\delta\leqslant d} X_\delta$ *we can state*

$$\left\| k - \Big(\widehat{\bigotimes_{1\leqslant\delta\leqslant d}} P_n^\delta \Big)(k) \right\|_\omega = O(\max_{1\leqslant\delta\leqslant d} \alpha_\delta(n))$$

for $n \to \infty$.

The importance of this theorem is given by the fact that for elements k belonging to the subspace $\widehat{\bigotimes}{}^{\eta}_{1\leqslant\delta\leqslant d} X_\delta$ one can get multivariate convergence statements of the same quality as in the one-dimensional case without restrictions concerning equicontinuity of the operator families $\{P_n^\delta\}_{n\in\mathbb{N}}$.

4. APPLICATIONS

Now we are going to consider several applications of Theorems 3, 4, 5, and 6.

(A) *Multivariate Cubic Spline Interpolation*

First we investigate the multivariate cubic periodic spline interpolation operators $S_{n_1 \cdots n_d}$. To this end we need corresponding results in the one-dimensional case which was treated by Cheney and Schurer [3, 4].

Let $C_*[0, 1]$ be the space of all continuous functions f on $[0, 1]$ satisfying $f(0) = f(1)$, and provide $C_*[0, 1]$ with the supremum norm. Then (see Cheney and Schurer [3]) the periodic cubic spline operator with respect to the equidistant nodes $\eta_\nu^{(n)} := \nu/n$ $(0 \leqslant \nu \leqslant n)$ has the following form:

$$S_n\colon C_*[0, 1] \ni f \mapsto S_n(f) = \sum_{\nu=0}^{n} f(\nu/n) \cdot s_\nu^{(n)} \in C_*[0, 1],$$

where the $s_\nu^{(n)}$ are periodic cubic spline functions associated with the nodes $\{\eta_0^{(n)},..., \eta_n^{(n)}\}$ satisfying $s_\nu^{(n)}(\eta_\mu^{(n)}) = \delta_{\mu\nu}$. Cheney and Schurer [3] proved that for the Chebyshev norm we have

$$\| f - S_n(f)\| \leqslant \| S_n \| \cdot \omega(f, 1/2n),$$

and, in addition, the estimation

$$\| S_n \| \leqslant \tfrac{1}{4}(1 + 3^{3/2}),$$

where ω is the modulus of continuity.

These results apply for multicubic periodic spline interpolation. In order to do this provide the nodes and the operators occurring above with indices δ $(1 \leqslant \delta \leqslant d)$. Then the d-dimensional periodic cubic spline operator has the form

$$S_{n_1 \cdots n_d} := \widehat{\bigotimes_{1 \leqslant \delta \leqslant d}} S_{n_\delta}^{\delta}\colon C_*\left(\mathop{\times}_{\delta=1}^{d} [0, 1]\right) \to C_*\left(\mathop{\times}_{\delta=1}^{d} [0, 1]\right).$$

Here $C_*(\times_{\delta=1}^{d} [0, 1])$ denotes the space of all functions $h \in C(\times_{\delta=1}^{d} [0, 1])$ that satisfy $h_\delta(0) = h_\delta(1)$ for $1 \leqslant \delta \leqslant d$ (in the notation as in Proposition 1). With the aid of Cheney and Schurer's result [3] and Theorem 4, we get for each $h \in C_*(\times_{\delta=1}^{d} [0, 1])$ for not necessarily coinciding $n_1,..., n_d$ the convergence

$$\lim_{(n_1,\ldots,n_d)\to\infty} \| h - S_{n_1 \cdots n_d}(h)\|_\epsilon = 0.$$

In addition, we can get the quantitative estimation:

THEOREM 7. *For any function $h \in C_*(\times_{\delta=1}^{d} [0, 1])$ the following estimation holds true*:

$$\| h - S_{n_1 \cdots n_d}(h)\|_\epsilon \leqslant \min_{\sigma \in \mathscr{S}_d} \sum_{\delta=1}^{d} \prod_{\nu=\delta+1}^{d} \| S_{n_{\sigma(\nu)}}^{\sigma(\nu)} \|_{[\sigma(\nu)]} \cdot \omega_{\sigma(\delta)}\left(h, \frac{1}{2n_{\sigma(\delta)}}\right).$$

Here the kth partial modulus of continuity ω_k of a function $f \in C(\mathsf{X}_{\delta=1}^{d} I_\delta)$ is defined by

$$\omega_k(f, t) := \sup_{\substack{x_\delta \in I_\delta \\ 1 \leqslant \delta \leqslant d \\ \delta \neq k}} \sup_{\substack{|x_k - \bar{x}_k| \leqslant t \\ x_k, \bar{x}_k \in I_k}} | f(x_1, ..., x_k, ..., x_d) - f(x_1, ..., \bar{x}_k, ..., x_d)|,$$

where $1 \leqslant k \leqslant d$, $t \in \mathbb{R}$, $t \geqslant 0$, and I_δ are compact intervals (*see* [22]).

Proof. Taking the notation of Proposition 1, we obtain with the aid of Theorem 5 and Proposition 1 the estimation

$$\begin{aligned}
&\| h - S_{n_1 \cdots n_d}(h) \|_\epsilon \\
&\leqslant \min_{\sigma \in \mathscr{S}_d} \sum_{\delta=1}^{d} \prod_{\nu=\delta+1}^{d} \| S_{n_{\sigma(\nu)}}^{\sigma(\nu)} \|_{[\sigma(\nu)]} \cdot \sup_{\substack{x_j \in [0,1] \\ 1 \leqslant j \leqslant d \\ j \neq \sigma(\delta)}} \| (id^{\sigma(\delta)} - S_{n_{\sigma(\delta)}}^{\sigma(\delta)})(h_{\sigma(\delta)}) \|_{\sigma(\delta)} \\
&\leqslant \min_{\sigma \in \mathscr{S}_d} \sum_{\delta=1}^{d} \prod_{\nu=\delta+1}^{d} \| S_{n_{\sigma(\nu)}}^{\sigma(\nu)} \|_{[\sigma(\nu)]} \cdot \sup_{\substack{x_j \in [0,1] \\ 1 \leqslant j \leqslant d \\ j \neq \sigma(\delta)}} \omega \left(h_{\sigma(\delta)}, \frac{1}{2n_{\sigma(\delta)}} \right) \\
&= \min_{\sigma \in \mathscr{S}_d} \sum_{\delta=1}^{d} \prod_{\nu=\delta+1}^{d} \| S_{n_{\sigma(\nu)}}^{\sigma(\nu)} \|_{[\sigma(\nu)]} \cdot \omega_{\sigma(\delta)} \left(h, \frac{1}{2n_{\sigma(\delta)}} \right). \quad \blacksquare
\end{aligned}$$

In this case of periodic cubic spline interpolation the corresponding interpolation operators are *equicontinuous*. Because of this fact our estimations in several variables are of the same quantitative behavior as corresponding one-dimensional assertions (up to some more complicated constants). If the sequence of interpolation operators is *not equicontinuous* then Theorem 5 yields *worse estimations in the several variables' case* than in one dimension because of the norms $\| P_{n_{\sigma(\nu)}}^{\sigma(\nu)} \|_{[\sigma(\nu)]}$ occurring in Theorem 5. We are going to point out this in the case of *multivariate Lagrange* interpolation. In addition, Theorem 6 enables us to get better results for certain functions lying in some subspaces.

(B) *Lagrange Interpolation in Several Variables*

Let the compact intervals $J_\delta = [-1, 1]$ $(1 \leqslant \delta \leqslant d)$ be given as well as the d infinite triangular matrices M_δ which we shall call nodal matrices (cf., [15]):

$$M_\delta := \{x_{\delta 0}^{(n)}, ..., x_{\delta n}^{(n)}\}_{n \geqslant 0} \qquad (1 \leqslant \delta \leqslant d),$$

where we have

(i) $-1 \leqslant x_{\delta\nu}^{(n)} \leqslant 1$ for $1 \leqslant \delta \leqslant d$, $0 \leqslant \nu \leqslant n$, $n = 0, 1, 2, ...,$

(ii) Fixed any δ and any $n \in \mathbb{N}_0$, then

$$x_{\delta\mu}^{(n)} \neq x_{\delta\nu}^{(n)} \qquad (\text{for } \mu \neq \nu)$$

holds true.

For $x_\delta \in J_\delta$ we define

$$\bar{\omega}_\delta^{(n)}(x_\delta) := \prod_{\nu=0}^{n} (x_\delta - x_{\delta\nu}^{(n)}),$$

$$v_{\delta\nu}^{(n)}(x_\delta) := 1 - \frac{\bar{\omega}_\delta^{(n)''}(x_{\delta\nu}^{(n)})}{\bar{\omega}_\delta^{(n)'}(x_{\delta\nu}^{(n)})} \cdot (x_\delta - x_{\delta\nu}^{(n)})$$

and

$$l_{\delta\nu}^{(n)}(x_\delta) := \frac{\bar{\omega}_\delta^{(n)}(x_\delta)}{\bar{\omega}_\delta^{(n)'}(x_{\delta\nu}^{(n)}) \cdot (x_\delta - x_{\delta\nu}^{(n)})}.$$

A nodal matrix M_δ is called ρ_δ-normal, if

$$v_{\delta\nu}^{(n)}(x_\delta) \geqslant \rho_\delta > 0$$

for all $x_\delta \in J_\delta$, all $n \in \mathbb{N}_0$, and all $\nu \in \{0,..., n\}$.

If we take $M_\delta = \{\xi_{\delta 0}^{(n)},..., \xi_{\delta n}^{(n)}\}_{n \geqslant 0}$, where $\{\xi_{\delta 0}^{(n)},..., \xi_{\delta n}^{(n)}\}$ is the set of zeros of the $(n+1)$th Chebyshev polynomial of the first kind, the latter is given on $[-1, 1]$ by $T_{n+1}(x) = \cos((n+1) \cdot \arccos x)$, then M_δ is $\frac{1}{2}$-normal. In the latter case, the nodes will be called Chebyshev nodes.

With the aid of the fundamental functions of Lagrange interpolation, $l_{\delta\nu}^{(n_\delta)}$, we define the δth Lagrange interpolation operator

$$L_{n_\delta}^{\delta}\colon C(J_\delta) \ni f \mapsto L_{n_\delta}^{\delta}(f) \in C(J_\delta)$$

by

$$L_{n_\delta}^{\delta}(f) := \sum_{\nu=0}^{n_\delta} f(x_{\delta\nu}^{(n_\delta)}) \cdot l_{\delta\nu}^{(n_\delta)}$$

for a given nodal matrix M_δ on J_δ. For ρ_δ-normal resp., Chebyshev nodes the following one dimensional estimates in the Chebyshev norm hold (see [15]):

$$\|f - L_{n_\delta}^{\delta}(f)\| = O\left(\frac{n_\delta^{1/2}}{n_\delta^{r_\delta+\alpha_\delta}}\right) \qquad (n_\delta \to \infty)$$

resp.,

$$\|f - L_{n_\delta}^{\delta}(f)\| = O\left(\frac{\log n_\delta}{n_\delta^{r_\delta+\alpha_\delta}}\right) \qquad (n_\delta \to \infty)$$

provided that $f \in C^{r_\delta}(J_\delta)$, the space of all r_δ-times continuously differentiable functions, and $f^{(r_\delta)} \in \mathrm{Lip}\, \alpha_\delta$ $(0 < \alpha_\delta \leqslant 1)$. In the multivariate case these results yield

THEOREM 8. *Denote by $L_{n_1 \cdots n_d}$ (resp., $\tilde{L}_{n_1 \cdots n_d}$) the d-dimensional Lagrange interpolation operator*

$$L_{n_1 \cdots n_d} := \widehat{\bigotimes_{1 \leqslant \delta \leqslant d}} L^{\delta}_{n_\delta} : C\left(\mathop{\times}_{\delta=1}^{d} J_\delta\right) \to C\left(\mathop{\times}_{\delta=1}^{d} J_\delta\right)$$

with respect to ρ_δ-normal nodes $(1 \leqslant \delta \leqslant d)$ (resp., Chebyshev nodes) in each component, and suppose $h \in C(\times_{\delta=1}^{d} J_\delta)$ possesses the partial derivatives $\partial^{r_\delta} h / \partial x_\delta^{r_\delta}$, which latter lie in $\mathrm{Lip}\, \alpha_\delta$ $(0 < \alpha_\delta \leqslant 1)$ for $1 \leqslant \delta \leqslant d$ (independent of the other variables), then the following estimations hold:

$$\| h - L_{n_1 \cdots n_d}(h) \|_\epsilon = O\left(\min_{\sigma \in \mathscr{S}_d} \sum_{\delta=1}^{d} \frac{(n_\delta^{1/2})^{\sigma(\delta)}}{n_\delta^{r_{\sigma(\delta)} + \alpha_{\sigma(\delta)}}} \right)$$

resp.,

$$\| h - \tilde{L}_{n_1 \cdots n_d}(h) \|_\epsilon = O\left(\min_{\sigma \in \mathscr{S}_d} \sum_{\delta=1}^{d} \frac{(\log n_\delta)^{\sigma(\delta)}}{n_\delta^{r_{\sigma(\delta)} + \alpha_{\sigma(\delta)}}} \right)$$

$$\text{for} \quad (n_1, \ldots, n_d) \to \infty.$$

This estimation turns out to be worse than the corresponding one-dimensional estimation above. For the subspace

$$\bigotimes_{1 \leqslant \delta \leqslant d}^{\epsilon} C(J_\delta) \subset \widehat{\bigotimes_{1 \leqslant \delta \leqslant d}^{\epsilon}} C(J_\delta) \cong C\left(\mathop{\times}_{\delta=1}^{d} J_\delta\right)$$

a theorem holds true which makes assertions on multivariate interpolation convergence of the same quality as the one-dimensional results. It is a consequence of Theorem 3.

THEOREM 9. *Let $h = \sum_{\kappa=1}^{k} f_{1\kappa} \otimes \cdots \otimes f_{d\kappa} \in \bigotimes_{1 \leqslant \delta \leqslant d}^{\epsilon} C(J_\delta)$ be given such that the functions $f_{\delta\kappa}$ are $r_{\delta\kappa}$-times differentiable and $f_{\delta\kappa}^{(r_{\delta\kappa})} \in \mathrm{Lip}\, \alpha_{\delta\kappa}$ $(0 < \alpha_{\delta\kappa} \leqslant 1)$. Then we have the following estimations for ρ_δ-normal (resp., Chebyshev) nodes:*

$$\| h - L_{n_1 \cdots n_d}(h) \|_\epsilon = O\left(\max_{\substack{1 \leqslant \kappa \leqslant k \\ 1 \leqslant \delta \leqslant d}} \frac{n_\delta^{1/2}}{n_\delta^{r_{\delta\kappa} + \alpha_{\delta\kappa}}} \right)$$

resp.,

$$\| h - \tilde{L}_{n_1 \cdots n_d}(h) \|_\epsilon = O\left(\max_{\substack{1 \leqslant \kappa \leqslant k \\ 1 \leqslant \delta \leqslant d}} \frac{\log n_\delta}{n_\delta^{r_{\delta\kappa} + \alpha_{\delta\kappa}}} \right) \quad \text{for} \quad (n_1, \ldots, n_d) \to \infty.$$

In some special situations, one can achieve the same order of convergence of interpolation in the multivariate case as in the one-dimensional case even if one does not restrict to the noncompleted tensor product spaces as in Theorem 9. In the following theorem we want to point out this fact for the multivariate Lagrange interpolation.

To this end we introduce the space $C^{p_1 \cdots p_d}(\mathsf{X}_{\delta=1}^d J_\delta)$ which consists of all functions whose derivatives $D^{\pi_1 \cdots \pi_d}$ exist and are continuous whenever $0 \leqslant \pi_\delta \leqslant p_\delta$ $(1 \leqslant \delta \leqslant d)$. Provide $C^{p_1 \cdots p_d}(\mathsf{X}_{\delta=1}^d J_\delta)$ with the norm of uniform convergence in these derivatives, i.e.,

$$\| h \|_{p_1 \cdots p_d} := \max_{\substack{1 \leqslant \delta \leqslant d \\ 0 \leqslant \pi_\delta \leqslant p_\delta}} \sup_{x_\delta \in J_\delta} | D^{\pi_1 \cdots \pi_d} h(x_1, ..., x_d) |.$$

Then Theorem 6 yields

THEOREM 10. *Let $L_{n_1 \cdots n_d}$ (resp., $\tilde{L}_{n_1 \cdots n_d}$) be the d-dimensional Lagrange interpolation operator,*

$$L_{n_1 \cdots n_d} : C\left(\mathop{\mathsf{X}}_{\delta=1}^{d} J_\delta\right) \to C\left(\mathop{\mathsf{X}}_{\delta=1}^{d} J_\delta\right),$$

with respect to ρ_δ-normal (resp., the Chebyshev) nodes in $J_\delta = [-1, 1]$ $(1 \leqslant \delta \leqslant d)$. Then for all $h \in C^{p_1 \cdots p_d}(\mathsf{X}_{\delta=1}^d J_\delta)$ the following estimations hold true:

$$\| h - L_{n_1 \cdots n_d}(h) \|_\epsilon = O\left(\max_{1 \leqslant \delta \leqslant d} \frac{n_\delta^{1/2}}{n_\delta^{p_\delta}} \right)$$

resp.,

$$\| h - \tilde{L}_{n_1 \cdots n_d}(h) \|_\epsilon = O\left(\max_{1 \leqslant \delta \leqslant d} \frac{\log n_\delta}{n_\delta^{p_\delta}} \right) \qquad \textit{for} \quad (n_1, ..., n_d) \to \infty.$$

Proof. By Treves [23], we have the isometry

$$C^{p_1 \cdots p_d}\left(\mathop{\mathsf{X}}_{\delta=1}^{d} J_\delta\right) \cong \mathop{\widehat{\bigotimes}^{\epsilon}}_{1 \leqslant \delta \leqslant d} C^{p_\delta}(J_\delta).$$

If for any $\delta \in \{1, 2, ..., d\}$, j_δ resp., $\hat{L}_{n_\delta}^\delta$ denote the mappings

$$j_\delta : C^{p_\delta}(J_\delta) \ni f \mapsto f \in C(J_\delta),$$

$$\hat{L}_{n_\delta}^\delta : C^{p_\delta}(J_\delta) \ni g \mapsto L_{n_\delta}^\delta(g) \in C(J_\delta),$$

then the following estimation holds true:

$$\| \hat{L}^{\delta}_{n_\delta} - j_\delta \| = \begin{cases} O\left(\dfrac{n_\delta^{1/2}}{n_\delta^{p_\delta}}\right), & \text{for } \rho_\delta\text{-normal nodes,} \\ O\left(\dfrac{\log n_\delta}{n_\delta^{p_\delta}}\right), & \text{for Chebyshev nodes,} \end{cases}$$

(for $n_\delta \to \infty$), where $\| \cdot \|$ is the corresponding operator norm. Indeed, with the aid of Jackson's theorems we have

$$\begin{aligned} \| j_\delta - \hat{L}^{\delta}_{n_\delta} \| &= \sup_{\|g\|_{p_\delta} \leqslant 1} \| j_\delta(g) - \hat{L}^{\delta}_{n_\delta}(g) \| \\ &= \sup_{\|g\|_{p_\delta} \leqslant 1} \| j_\delta(g) - L^{\delta}_{n_\delta}(g) \| \leqslant \sup_{\|g\|_{p_\delta} \leqslant 1} (\| L^{\delta}_{n_\delta} \| + 1) \cdot E_{n_\delta}(g) \\ &\leqslant K \cdot (\| L^{\delta}_{n_\delta} \| + 1) \frac{1}{n_\delta^{p_\delta}} =: A_i , \end{aligned}$$

($i = 1$ for ρ_δ-normal nodes, $i = 2$ for Chebyshev nodes). Here $E_{n_\delta}(g)$ denotes the approximation constant of g with respect to the Chebyshev-norm, and $\| \cdot \|_{p_\delta}$ is the norm of uniform convergence of the derivatives of order π_δ ($0 \leqslant \pi_\delta \leqslant p_\delta$).

In the case of ρ_δ-normal nodes we have

$$A_1 \leqslant K_1 \cdot \frac{n_\delta^{1/2}}{n_\delta^{p_\delta}},$$

and in the second case

$$A_2 \leqslant K_2 \cdot \frac{\log n_\delta}{n_\delta^{p_\delta}}$$

holds true. Hence with the aid of Theorem 6 the proof is established. ∎

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